Statistical Mechanics of systems with long range interactions

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Abstract. Recent theoretical studies of statistical mechanical properties of systems with long range interactions are briefly reviewed. In these systems the interaction potential decays with a rate slower than $1/r^d$ at large distances r in d dimensions. As a result, these systems are non-additive and they display unusual thermodynamic and dynamical properties which are not present in systems with short range interactions. In particular, the various statistical mechanical ensembles are not equivalent and the microcanonical specific heat may be negative. Long range interactions may also result in breaking of ergodicity, making the maximal entropy state inaccessible from some regions of phase space. In addition, in many cases long range interactions result in slow relaxation processes, with time scales which diverge in the thermodynamic limit. Various models which have been found to exhibit these features are discussed.

Keywords: long range interactions, inequivalence of ensembles, slow relaxation, quasistationary states, ergodicity breaking
PACS: 05.20.Gg, 05.50.+q, 05.70.Fh

1. INTRODUCTION

Long range forces are rather common in nature. These forces are typically derived from two body potentials which at large distance, r, decay as $1/r^s$ with $s \leq d$ in d dimensions. Examples include self gravitating systems $(s = 1)$ [1, 2], dipolar ferroelectrics and ferromagnets $(s = 3)$ [3], non-neutral plasmas $(s = 1)$ [4], two dimensional geophysical vortices which interact via a weak, logarithmically decaying, potential $(s = 0)$ [2], charged particles interacting via their mutual electromagnetic fiels, such as in free electron laser [5] and many others (for recent reviews see [6]). As a result of the long range nature of the interactions, these systems are non-additive, and the energy of homogeneously distributed particles in a volume V scales super-linearly with the volume, as $V^{1+\sigma}$, with $\sigma = 1 - s/d \geq 0$. The lack of additivity leads to many unusual properties, both thermal and dynamical, which are not present in the more commonly studied systems with short range interactions. For example, as has first been pointed out by Antonov [7] and later elaborated by Lynden-Bell [8, 9], Thirring [10, 11] and others, the entropy S needs not be a concave function of the energy E, yielding negative specific heat within the microcanonical ensemble. Since specific heat is always positive when calculated within the canonical ensemble, this indicates that the two ensembles need not be equivalent. Recent studies have suggested the inequivalence of ensembles is particularly manifested whenever a model exhibits a first order transition within the canonical ensemble [12, 13]. Similar ensemble inequivalence between canonical and grand canonical ensembles has also been discussed [14].

Studies of the relaxation processes is systems with long range interactions in some models have shown that the relaxation of thermodynamically unstable states to the stable equilibrium state may be unusually slow, with a characteristic time which diverges with the number of particles, N, in the system [15, 16, 17, 18, 19, 20]. This, too, is in contrast with relaxation processes in systems with short range interactions, in which the relaxation time does not scale with N. As a result long lived quasi-stationary states (QSS) have been observed in some models, which in the thermodynamic limit, do not relax to the equilibrium state. Non-additivity has been found to result, in many cases, in breaking of ergodicity. Here phase space is divided into disjoint domains separated by finite gaps in macroscopic quantities, such as the total magnetization in magnetic systems [20, 21, 22, 23, 24, 25, 26]. Within local dynamics, these systems are thus trapped in one of the domains.

Typically, the entropy, S, which is measured by the number of ways N particles with total energy E may be distributed in a volume V, scales linearly with the volume. This is irrespective of whether or not the interactions in the system are long ranged. On the other hand, in systems with long range interactions, the energy scales super-linearly with the volume. Thus, in the thermodynamic limit, the free energy $F = E - TS$ is dominated by the energy at any finite temperature T, suggesting that the entropy may be neglected altogether. This would result in trivial thermodynamics. However, in many real cases, when systems of finite size are considered, the temperature could be sufficiently high so that the entropic term in the free energy, TS, becomes comparable to the energy E. In such cases the entropy may not be neglected and the thermodynamics is non trivial. This is the case in some self gravitating systems such as globular clusters (see, for example [2]). In order to theoretically study this limit, it is convenient to rescale the energy by a factor $V^{-\sigma}$ (or alternatively, to rescale the temperature by a factor V^{σ}), making the energy and the entropy contribution to the free energy of comparable magnitude. This is known as the Kac prescription [27]. While systems described by this rescaled energy are extensive, they are non-additive in the sense that the energy of two isolated sub-systems is not equal to their total energy when they are combined together and are allowed to interact.

A special case is that of dipolar ferromagnets, where the interaction scales as $1/r^3$ ($\sigma = 0$). In this borderline case between long and short range interactions, the energy depends on the shape of the sample. It is well known that for ellipsoidal magnets, the contribution of the long distance part of the dipolar interaction leads to a mean-field type term in the energy. This results in an effective Hamiltonian $H \rightarrow H - DM^2/N$, where M is the magnetization of the system and D is a shape dependent coefficient known as the demagnetization factor. In this Hamiltonian, the long range interaction between dipoles becomes independent of their distance, making it particularly convenient for theoretical studies [28].

Non-additivity is a feature which is not limited to systems with long range interactions. In fact finite systems with short range interactions, in which surface and bulk energies are comparable, are also non-additive. Features such as negative specific heat in small systems (e.g. clusters of atoms) have been discussed a number of studies [29, 30, 31].

In the present paper we briefly review recent theoretical studies of systems with long range interactions where such properties have been explored. In Section (2) general

considerations are presented, arguing for some of the unusual properties of systems with long range interactions. In Section (3) some features of canonical and microcanonical phase diagrams are discussed within a recently studied Ising model with mean-field type interaction. Ergodicity breaking is discussed in Section (4), and slow relaxation processes, as observed in a number of models, are discussed in Section (5). A summary and general outlook is finally given in Section (6).

2. GENERAL CONSIDERATIONS

We start by presenting some general considerations concerning thermodynamic properties of systems with long range interactions. In particular we argue that in addition to negative specific heat, or non-concave entropy curve, which could be realized in the microcanonical ensemble, this ensemble also yields discontinuity in temperature whenever a first order transition takes place.

Consider the non-concave curve of Fig. (1). For a system with short range interactions, this curve cannot represent the entropy $S(E)$. The reason is that due to additivity, the system represented by this curve is unstable in the energy interval $E_1 < E < E_2$. Entropy can be gained by phase separating the system into two subsystems corresponding to E_1 and E_2 keeping the total energy fixed. The average energy and entropy densities in the coexistence region is given by the weighted average of the corresponding densities of the two coexisting systems. Thus the correct entropy curve in this region is given by the common tangent line, resulting in an overall concave curve. However, in systems with long range interactions, the average energy density of two coexisting subsystems is not given by the weighted average of the energy density of the two subsystems. Therefore, the non-concave curve of Fig. (1) could, in principle, represent an entropy curve of a stable system, and phase separation need not take place. This results in negative specific heat. Since within the canonical ensemble specific heat is non-negative, the microcanonical and canonical ensembles are not equivalent. The above considerations suggest that the inequivalence of the two ensembles is particularly manifested whenever a coexistence of two phases is found within the canonical ensemble.

Another feature of systems with long range interactions is that within the microcanonical ensemble, first order phase transitions involve discontinuity of temperature. To demonstrate this point consider, for example, a magnetic system which undergoes a phase transition from a paramagnetic to a magnetically ordered phase. Let M be the magnetization and $S(M,E)$ be the entropy of the system for a given magnetization and energy. A typical entropy vs magnetization curve for a given energy close to a first order transition is given in Fig. (2). It exhibits three local maxima, one at $M = 0$ and two other degenerate maxima at $\pm M_0$. At energies where the paramagnetic phase is stable, one has $S(0,E) > S(\pm M_0, E)$. In this phase the entropy is given by $S(0,E)$ and the temperature is obtained by $1/T = dS(0,E)/dE$. On the other hand at energies where the magnetic phase is stable, the entropy is given by $S(M_0,E)$ and the temperature is $1/T = dS(M_0,E)/dE$. At the first order transition point, where $S(0,E) = S(\pm M_0, E)$, the two derivatives are generically not equal, resulting in a temperature discontinuity. A typical entropy vs energy curve is given in Fig. (3).

Systems with long range interactions are more likely to exhibit breaking of ergodicity

due to their non-additive nature. This may be argued on rather general grounds. In systems with short range interactions, the domain in the phase space of extensive thermodynamic variables, such as energy, magnetization, volume etc., is convex. Let $\vec{X}$ be a vector whose components are the extensive thermodynamic variables over which the systems is defined. Suppose that there exist microscopic configurations corresponding to two points $\vec{X}_1$ and $\vec{X}_2$ in this phase space. As a result of the additivity property of systems with short range interactions, there exist microscopic configurations corresponding to any intermediate point between $\vec{X}_1$ and $\vec{X}_2$. Such microscopic configurations may be constructed by combining two appropriately weighted subsystems corresponding to $\vec{X}_1$ and $\vec{X}_2$, making use of the fact that for sufficiently large systems, surface terms do not contribute to bulk properties. Since systems with long range interactions are non-additive, such interpolation is not possible, and intermediate values of the extensive variables are not necessarily accessible. As a result the domain in the space of extensive variables over which a system is defined needs not be convex. When there exists a gap in phase space between two points corresponding to the same energy, local energy conserving dynamics cannot take the system from one point to the other and ergodicity is broken.

These and other features of canonical and microcanonical phase diagrams are explored in the following sections by considering specific models.

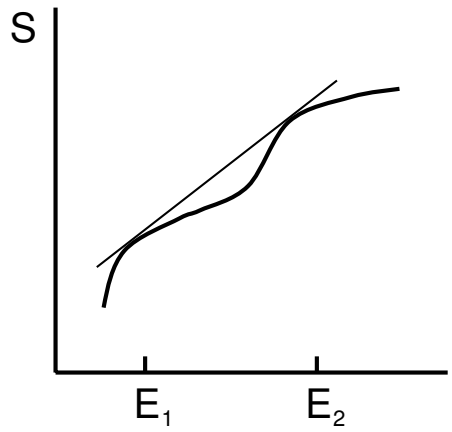

FIGURE 1. A non-concave entropy curve, which for additive systems is made concave by the common tangent line. In systems with long range interactions, the non-concave curve may represent the actual entropy of the system, yielding negative specific heat.

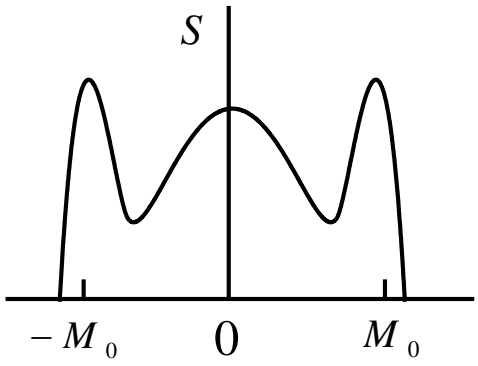

FIGURE 2. A typical entropy vs magnetization curve of a magnetic system with long range interactions near a first order transition at a given energy. As the energy varies the heights of the peaks change and a first order transition is obtained at the energy where the peaks are of equal height.

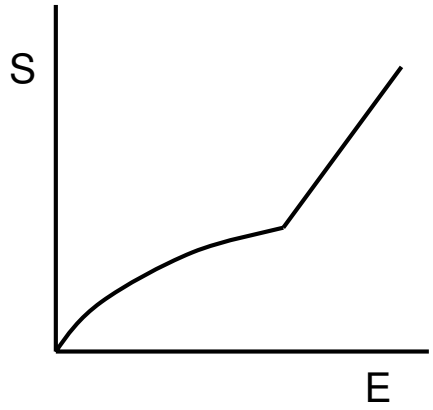

FIGURE 3. A typical entropy vs energy curve for a system with long range interactions exhibiting a first order transition. The slope discontinuity at the transition results in a temperature discontinuity.

3. PHASE DIAGRAMS OF MODELS WITH LONG RANGE INTERACTIONS

In order to obtain better insight into the thermodynamic behavior of systems with long range interactions it is instructive to analyze phase diagrams of representative models. A particularly convenient class of models is that where the long range part of the interaction is of mean-field type. In such models $\sigma = 0$, and as pointed out above, they may be applied to study dipolar ferromagnets [28]. The insight obtained from studies of these models may, however, be relevant for other systems with $\sigma > 0$, since the main feature

of these models, namely non-additivity, is shares by all models with $\sigma \geq 0$.

In recent studies both the canonical and microcanonical phase diagrams of some spin models with mean-field type long range interactions have been analyzed. Examples include discrete spin models such as the Blume-Emery-Griffiths model [12, 13] and the Ising model with long and short range interactions [20] as well as continuous spin models of XY type [32, 33]. These models are simple enough so that their thermodynamic properties can be evaluated in both ensembles. The common feature of these models is that their phase diagrams exhibit first and second order transition lines. In has been found that in all cases, the canonical and microcanonical phase diagrams differ from each other in the vicinity of the first order transition line. A classification of possible types of inequivalent canonical and microcanonical phase diagrams in systems with long range interactions is given in [34]. In what follows we discuss in some detail the thermodynamics of one model, namely, the Ising model with long and short range interactions [20].

Consider an Ising model defined on a ring with N sites. Let $S_i = \pm 1$ be the spin variable at site $i = 1, \ldots, N$. The Hamiltonian of the systems is composed of two interaction terms and is given by

$$H = -\frac{K}{2}\sum_{i=1}^{N}(S_i S_{i+1} - 1) - \frac{J}{2N}\left(\sum_{i=1}^{N} S_i\right)^2. \tag{1}$$

The first term is a nearest neighbor coupling which could be either ferromagnetic $(K > 0)$ or antiferromagnetic $(K < 0)$. On the other hand the second term is ferromagnetic, $J > 0$, and it corresponds to long range, mean-field type interaction. The reason for considering a ring geometry for the nearest neighbor coupling is that this is more convenient for carrying out the microcanonical analysis. Similar features are expected to take place in higher dimensions as well.

The canonical phase diagram of this model has been analyzed some time ago [35, 36]. The ground state of the model is ferromagnetic for $K > -J/2$ and is antiferromagnetic for $K < -J/2$. Since the system is one dimensional, and since the long range interaction term can only support ferromagnetic order, it is clear that for $K < -J/2$ the system is disordered at any finite temperature, and no phase transition takes place. However, for $K > -J/2$ one expects ferromagnetic order at low temperatures. Thus a phase transition takes place at some temperature to a paramagnetic, disordered phase (see Fig. 4). For large K the transition was found to be continuous, taking place at temperature given by

$$\beta = e^{-\beta K}. \tag{2}$$

Here $\beta = 1/T$, $J = 1$ is assumed for simplicity, and $k_B = 1$ is taken for the Boltzmann constant. The transition becomes first order for $K < K_{CTP}$, with a tricritical point located at an antiferromagnetic coupling $K_{CTP} = -\ln 3/2\sqrt{3} \simeq -0.317$. As usual, the first order line has to be evaluated numerically. The first order line intersects the $T = 0$ axis at $K = -1/2$. The (K, T) phase diagram is given in Fig. (4).

Let us now analyze the phase diagram of the model within the microcanonical ensemble [20]. To do this one has to calculate the entropy of the system for given magnetization

and energy. Let

$$U = -\frac{1}{2}\sum_i (S_i S_{i+1} - 1) \tag{3}$$

be the number of antiferromagnetic bonds in a given configuration characterized by N_+ up spins and N_- down spins with $N_+ + N_- = N$. One would like to evaluate the number of microscopic configurations corresponding to (N_+, N_-, U). Such configurations are composed of $U/2$ segments of up spins which alternate with the same number of segments of down spins, where the total number of up (down) spins is N_+ (N_-). The number of ways of dividing N_+ spins into $U/2$ groups is

$$\left(\begin{array}{c} N_+ - 1 \\ U/2 - 1 \end{array} \right) , \tag{4}$$

with a similar expression for the down spins. To leading order in N, the number of configurations corresponding to (N_+, N_-, U) is given by

$$\Omega(N_+, N_-, U) = \left(\begin{array}{c} N_+ \\ U/2 \end{array} \right) \left(\begin{array}{c} N_- \\ U/2 \end{array} \right) . \tag{5}$$

Note that a multiplicative factor of order N has been neglected in this expression, since only exponential terms in N contribute to the entropy. This factor corresponds to the number of ways of placing the U ordered segments on the lattice. Expressing N_+ and N_- in terms of the number of spins, N, and the magnetization, $M = N_+ - N_-$, and denoting $m = M/N$, $u = U/N$ and the energy per spin $\varepsilon = E/N$, one finds that the entropy per spin, $s(\varepsilon, m) = \frac{1}{N} \ln \Omega$, is given in the thermodynamic limit by

$$\begin{aligned} s(\varepsilon, m) &= \frac{1}{2}(1+m)\ln(1+m) + \frac{1}{2}(1-m)\ln(1-m) \\ &- u \ln u - \frac{1}{2}(1+m-u)\ln(1+m-u) \\ &- \frac{1}{2}(1-m-u)\ln(1-m-u) , \end{aligned} \tag{6}$$

where u satisfies

$$\varepsilon = -\frac{J}{2} m^2 + Ku . \tag{7}$$

By maximizing $s(\varepsilon, m)$ with respect to m one obtains both the spontaneous magnetization $m_s(\varepsilon)$ and the entropy $s(\varepsilon) \equiv s(\varepsilon, m_s(\varepsilon))$ of the system for a given energy ε.

In order to analyze the microcanonical phase transitions corresponding to this entropy we expand s in powers of m,

$$s = s_0 + Am^2 + Bm^4 . \tag{8}$$

Here the zero magnetization entropy is

$$s_0 = -\frac{\varepsilon}{K} \ln \frac{\varepsilon}{K} - \left(1 - \frac{\varepsilon}{K}\right) \ln \left(1 - \frac{\varepsilon}{K}\right) , \tag{9}$$

the coefficient A is given by

$$A = \frac{1}{2}\left[\frac{1}{K}\ln\left(\frac{K-\varepsilon}{\varepsilon}\right) - \frac{\varepsilon}{K-\varepsilon}\right] , \tag{10}$$

and B is another energy dependent coefficient which can be easily evaluated. In the paramagnetic phase both A and B are negative so that the $m = 0$ state maximizes the entropy. At the energy where A vanishes, a continuous transition to the magnetically ordered state takes place. Using the thermodynamic relation for the temperature

$$\frac{1}{T} = \frac{ds}{d\varepsilon} , \tag{11}$$

the caloric curve in the paramagnetic phase is found to be

$$\frac{1}{T} = \frac{1}{K}\ln\frac{K-\varepsilon}{\varepsilon} . \tag{12}$$

This expression is also valid at the critical line where $m = 0$. Therefore, the critical line in the (K,T) plane may be evaluated by taking $A = 0$ and using (12) to express ε in terms of T. One finds that the expression for the critical line is the same as that obtained within the canonical ensemble, (2).

The transition is continuous as long as B is negative, where the $m = 0$ state maximizes the entropy. The transition changes its character at a microcanonical trictitical point where $B = 0$. This takes place at $K_{MTP} \simeq -0.359$, which may be computed analytically using the expression for the coefficient B. The fact that $K_{MTP} < K_{CTP}$ means that while the microcanonical and canonical critical lines coincide up to K_{CTP}, the microcanonical line extends beyond this point into the region where, within the canonical ensemble, the model is magnetically ordered (see Fig. (4)). In this region the microcanonical specific heat is negative. For $K < K_{MTP}$ the microcanonical transition becomes first order, and the transition line has to be evaluated numerically by maximizing the entropy. As discussed in the previous section, such a transition is characterized by temperature discontinuity. The shaded region in the (K,T) phase diagram of Fig. (4) indicates an inaccessible domain resulting from the temperature discontinuity.

The main features of the phase diagram given in Fig. (4) are not peculiar to the Ising model defined by the Hamiltonian (1), but are expected to be valid for any system in which a continuous line changes its character and becomes first order at a tricritical point. In particular, the lines of continuous transition are expected to be the same in both ensembles up to the canonical tricritical point. The microcanonical critical line extends beyond this point into the ordered region of the canonical phase diagram, yielding negative specific heat. When the microcanonical tricritical point is reached, the transition becomes first order, characterized by a discontinuity of the temperature. These features have been found in studies of other discrete spin models such as the spin-1 Blume-Emery-Griffiths model [12, 13]. They have also been found in continuous spin models such as the XY model with two- and four-spin mean-field like ferromagnetic interaction terms [32], and in an XY model with long and short range, mean-field type, interactions [33].

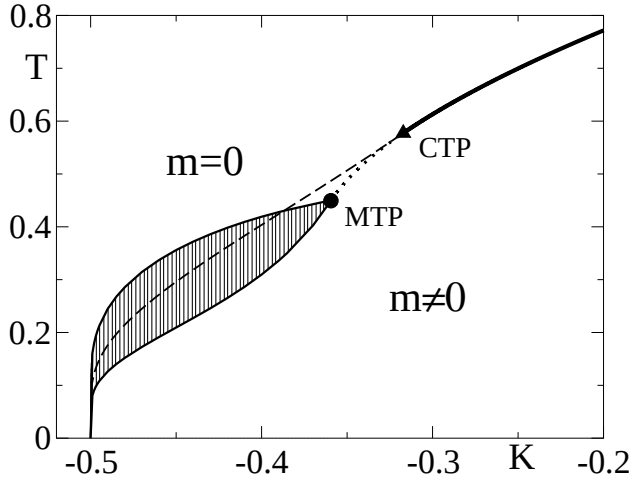

FIGURE 4. The (K,T) phase diagrams of the model (1) within the canonical and microcanonical ensembles. In the canonical ensemble the large K transition is continuous (bold solid line) down to the trictitical point CTP where it becomes first order (dashed line). In the microcanonical ensemble the continuous transition coincides with the canonical one at large K (bold line). It persists at lower K (dotted line) down to the tricritical point MTP where it turns first order, with a branching of the transition line (solid lines). The region between these two lines (shaded area) is not accessible.

4. ERGODICITY BREAKING

Ergodicity breaking in models with long range interactions has recently been explicitly demonstrated in a number of models such as a class of anisotropic XY models [22, 23], discrete spin Ising models [20], mean-field ϕ^4 models [24, 25] and isotropic XY models with four-spin interactions [26]. Here we outline a demonstration of this feature for the Ising model with long and short range interactions defined in the previous section [20].

Let us consider the Hamiltonian (1), and take, for simplicity, a configuration of the spins with $N_+ > N_-$. The local energy U is, by definition, non-negative. It also has an upper bound which, for the case $N_+ > N_-$, is $U \leq 2N_-$. This upper bound is achieved when the negative spins are isolated, each contributing two negative bonds to the energy. Thus $0 \leq u \leq 1-m$. Combining this with (7) one finds that for positive m the accessible states have to satisfy

$$\begin{aligned} & m \leq \sqrt{-2\varepsilon} \quad , \quad m \geq m_+ \quad , \quad m \leq m_- \\ & \text{with } m_\pm = -K \pm \sqrt{K^2 - 2(\varepsilon - K)} \, . \end{aligned} \tag{13}$$

Similar restrictions exist for negative m. These restrictions yield the accessible magnetization domain shown in Fig. (5) for $K = -0.4$.

The fact that the accessible magnetization domain is not convex results in nonergodicity. At a given, sufficiently low energy, the accessible magnetization domain is composed of two intervals with large positive and large negative magnetization, respectively. Thus starting from an initial condition which lies within one of these intervals, local dynamics, to be discussed in the next section, is unable to move the system to the other accessi-

ble interval, and ergodicity is broken. At intermediate energy values another accessible magnetization interval emerges near the $m = 0$ state and three disjoint magnetization intervals are available. When the energy is increased the the three intervals join together and the model becomes ergodic.

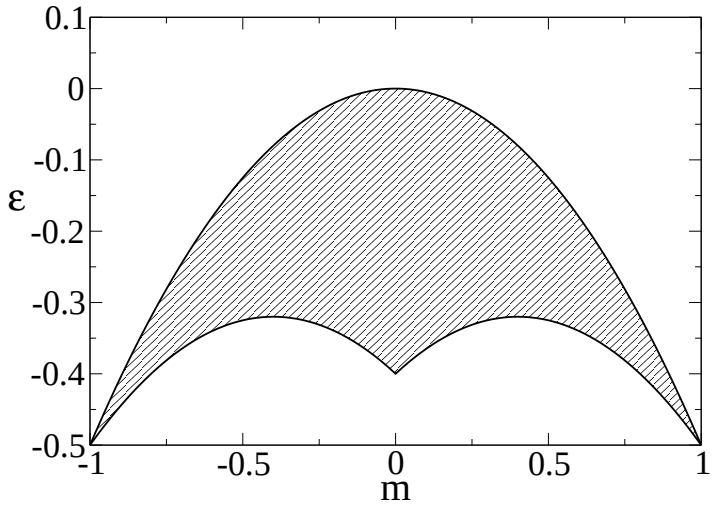

FIGURE 5. Accessible region in the (m, ε) plane (shaded area) of the Hamiltonian (1) with $K = -0.4$. At low energies, the accessible domain is composed of two disjoint magnetization intervals and at intermediate energies three such intervals exist, yielding ergodicity breaking. At higher energies the three intervals join together and ergodicity is restored.

5. SLOW RELAXATION

In systems with short range interactions the relaxation from a thermodynamically unstable state is typically a fast process. For example, in a magnetic, Ising like system, starting with a magnetically disordered state at a low temperature, where the stable state is the ordered one, the system will locally order in short time. This leads to a domain structure in which the system is divided into magnetically up and down domains of some typical size. The domains forming process is fast in the sense that its characteristic time does not scale with the system size. This domain structure is formed by fluctuations, when a locally ordered region reaches a critical size for which the loss its surface free energy is compensated by the gain in its bulk free energy. This critical size is independent of the system size, leading to a finite relaxation time. Once the domain structure is formed it exhibits a coarsening process in which the domains grow in size while their number is reduced. This process, which is typically slow, eventually leads to the ordered equilibrium state of the system.

This is very different from what happens in systems with long range interactions. Here the initial relaxation from a thermodynamically unstable state need not be fast and it could take place over a time scale which diverges with the system size. The reason is that in the case of long range interactions one cannot define a critical size of an ordered

domain, since the bulk and surface energies of a domain are of the same order. It is thus of great interest to study relaxation processes in systems with long range interactions and to explore the types of behavior which might be encountered. In principle the relaxation process may depend on the nature and symmetry of the order parameter, say, whether it is discrete, Ising like, or one with a continuous symmetry such as the XY model. It may also depend on the dynamical process, whether it is stochastic or deterministic. In this section we briefly review some recent results obtained in studies of the dynamics of some models with long range interactions.

We start by considering the Ising model with long and short range interactions defined in section (3). The relaxation processes in this model have recently been studied in [20]. Since Ising models do not have intrinsic dynamics, the common dynamics one uses in studying them is the Monte Carlo (MC) dynamics, which simulates the stochastic coupling of the model to a thermal bath. If one is interested in studying the dynamics of an isolated system, one has to resort to the microcanonical MC algorithm developed by Creutz [37] some time ago. According to this algorithm a demon with energy $E_d \geq 0$ is allowed to exchange energy with the system. One starts with a system with energy E and a demon with energy $E_d = 0$. The dynamics proceeds by selecting a spin at random and attempting to flip it. If, as a result of the flip, the energy of the system is reduced, the flip is carried out and the excess energy is transferred to the demon. On the other hand if the energy of the system increases as a result of the attempted flip, the energy needed is taken from the demon and the move is accepted. In case the demon does not have the necessary energy the move is rejected. After sufficiently long time and for large system size, N, the demon's energy will be distributed according to the Boltzmann distribution $\exp(-E_d/k_BT)$, where T is the temperature of the system with energy E. Thus, by measuring the energy distribution of the demon one obtains the caloric curve of the system. Note that as long as the entropy of the system is an increasing function of its energy, the temperature is positive and the average energy of the demon is finite. The demon's energy is thus negligibly small compared with the energy of the system, which scales with its size. The energy of the system at any given time is $E - E_D$, and it exhibits fluctuations of finite width at energies just below E.

In applying the microcanonical MC dynamics to models with long range interactions, one should note that the Boltzmann expression for the energy distribution of the demon is valid only in the large N limit. To next order in N one has

$$P(E_D) \sim \exp\left(-E_D/T - E_D^2/2C_VT^2\right) , \tag{14}$$

where $C_V = O(N)$ is the system's specific heat. In systems with short range interactions, the specific heat is non-negative and thus the next to leading term in the distribution function is a stabilizing factor which may be neglected for large N. On the other hand, in systems with long range interactions, C_V may be negative in some regions of the phase diagram, and on the face of it, the next to leading term may destabilize the distribution function. However the next to leading term is small, of order $O(1/N)$, and it is straightforward to argue that as long as the entropy is an increasing function of the energy, the next to leading term does not destabilize the distribution. The Boltzmann distribution for the energy of the demon is thus valid for large N.

Using the microcanonical MC algorithm, the dynamics of the model (1) has been studied in detail [20]. Breaking of ergodicity in the region in the (K,ε) plane where it is expected to take place has been observed.

The microcanonical MC dynamics has also been applied to study the relaxation process of thermodynamically unstable states. It has been found that starting with a zero magnetization state at energies where this state is a local minimum of the entropy, the model relaxes to the equilibrium, magnetically ordered, state on a time scale which diverges with the system size as $\ln N$. The divergence of the relaxation time is a direct result of the long range interactions in the model.

The logarithmic divergence of the relaxation time may be understood by considering the Langevin equation which corresponds to the dynamical process. The equation for the magnetization m is

$$\frac{\partial m}{\partial t} = \frac{\partial s}{\partial m} + \xi(t) \ , \ <\xi(t)\xi(t')> = D\delta(t-t') \tag{15}$$

where $\xi(t)$ is the usual white noise term. The diffusion constant D scales as $D \sim 1/N$. This can be easily seen by considering the non-interacting case in which the magnetization evolves by pure diffusion where the diffusion constant is known to scale in this form. Since we are interested in the case of a thermodynamically unstable $m=0$ state, which corresponds to a local minimum of the entropy, we may, for simplicity, consider an entropy function of the form

$$s(m) = am^2 - bm^4 \tag{16}$$

with a and b non-negative parameters. In order to analyze the the relaxation process we consider the corresponding Fokker-Planck equation for the probability distribution $P(m,t)$ of the magnetization at time t. It takes the form

$$\frac{\partial P(m,t)}{\partial t} = D\frac{\partial^2 P(m,t)}{\partial m^2} - \frac{\partial}{\partial m}\left(\frac{\partial s}{\partial m}P(m,t)\right) \ , \tag{17}$$

This equation could be viewed as describing the motion of a particle whose coordinate, m, carries out an overdamped motion in a potential $-s(m)$ at temperature $T=D$. In order to probe the relaxation process from the $m=0$ state it is sufficient to consider the entropy (16) with $b=0$. With the initial condition for the probability distribution $P(m,0)=\delta(m)$, the large time asymptotic distribution is found to be [38]

$$P(m,t) \sim \exp\left[-\frac{ae^{-at}m^2}{D}\right] . \tag{18}$$

This is a Gaussian distribution whose width grows with time. Thus, the relaxation time from the unstable state, τ_{us}, which corresponds to the width reaching a value of $O(1)$, satisfies

$$\tau_{us} \sim -\ln D \sim \ln N \ . \tag{19}$$

The logarithmic divergence with N of the relaxation time seems to be independent of the nature of the dynamics. Similar behavior has been found when the model (1) has been studied within the Metropolis-type canonical dynamics at fixed temperature [20].

The relaxation process from a metastable state (rather than an unstable state discussed above) has been studied rather extensively in the past. Here the entropy has a local maximum at $m = 0$, while the global maximum is obtained at some $m \neq 0$. As one would naively expect, the relaxation time from the metastable $m = 0$ state, τ_{ms}, is found to grow exponentially with N [20]

$$\tau_{ms} \sim e^{N\Delta s} \ . \tag{20}$$

The entropy barrier corresponding to the non-magnetic state, Δs, is the difference in entropy between that of the $m = 0$ state and the entropy at the local minimum separating it from the stable equilibrium state. Such exponentially long relaxation times are expected to take place independently of the nature of the order parameter or the type of dynamics (whether it is stochastic or deterministic). This has been found in the past in numerous studies of canonical, Metropolis-type dynamics, of the Ising model with mean-field interactions [39], in deterministic dynamics of the XY model [40] and in models of gravitational systems [41].

A different, rather intriguing, type of relaxation process has been found in studies of the Hamiltonian dynamics of the XY model with mean-field interactions [15, 16, 17, 18, 19]. This model has been termed the Hamiltonian Mean Field (HMF) model. In this model, some non-equilibrium quasi-stationary states have been identified, whose relaxation time grows as a power of the system size, N, for some energy interval. This non-equilibrium stationary state (which becomes a steady state in the thermodynamic limit) exhibit some interesting properties such as anomalous diffusion which have been extensively studied ([17, 18, 19, 42]). At other energy intervals the relaxation process has been found to be much faster, with a relaxation time which grows as $\ln N$ [43]. In what follows we briefly outline the main results obtained for the HMF model and for some generalizations of it.

The HMF model is defined on a lattice with each site occupied by an XY spin of unit length. The Hamiltonian takes the form

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} \left[1 - \cos(\theta_i - \theta_j)\right] \ , \tag{21}$$

where θ_i and p_i are the phase and momentum of the ith particle, respectively. In this model the interaction is mean-field like. The model exhibits a continuous transition at a critical energy $\varepsilon_c = 3/4$ from a paramagnetic state at high energies to a ferromagnetic state at low energies. Within the Hamiltonian dynamics, the equations of motion of the dynamical variables are

$$\frac{d\theta_i}{dt} = p_i \ , \qquad\qquad \frac{dp_i}{dt} = -m_x \sin\theta_i + m_y \cos\theta_i \ , \tag{22}$$

where m_x and m_y are the components of the magnetization density

$$\vec{m} = \left(\frac{1}{N} \sum_{i=1}^{N} \cos\theta_i, \frac{1}{N} \sum_{i=1}^{N} \sin\theta_i \right) \ . \tag{23}$$

The Hamiltonian dynamics obviously conserves both energy and momentum. A typical initial configuration for the non-magnetic state is taken as the one where the phase variables are uniformly and independently distributed in the interval $\theta_i \in [-\pi, \pi]$. A particularly interesting case is that where the initial distribution of the momenta is uniform in an interval $[-p_0, p_0]$. This has been termed the waterbag distribution. For such phase and momentum distributions the initial energy density is given by $\varepsilon = p_0^2/6 - 1/2$.

Extensive numerical studies of the relaxation of the non-magnetic state with the waterbag initial distribution have been carried out. It has been found that at an energy interval just below ε_c this state is quasi-stationary, in the sense that the magnetization fluctuates around its initial value for some time τ_{qs} before it switches to the non-vanishing equilibrium value. This characteristic time has been found to scale as [18, 19]

$$\tau_{qs} \sim N^{\gamma} \tag{24}$$

with $\gamma \simeq 1.7$.

A very useful insight into the dynamics of the HMF model is provided by analyzing the evolution of the probability distribution of the phase and momentum variables, $f(\theta, p, t)$, within the Vlasov equation approach [19]. It has been found that in the energy interval $\varepsilon^* < \varepsilon < \varepsilon_c$, with $\varepsilon^* = 7/12$, the waterbag distribution is linearly stable. It is unstable for $\varepsilon < \varepsilon^*$. In this interval the following growth law for the magnetization $m = \sqrt{m_x^2 + m_y^2}$ has been found [43]:

$$m(t) \sim \frac{1}{\sqrt{N}} e^{\Omega t} \ , \tag{25}$$

where

$$\Omega = \sqrt{6(\varepsilon^* - \varepsilon)} \ . \tag{26}$$

The robustness of the quasi-stationary state to various perturbations has been explored in a number of studies. The anisotropic HMF model has recently been shown to exhibit similar relaxation processes as the HMF model itself [43]. The anisotropic HMF model is defined by the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} \left[1 - \cos(\theta_i - \theta_j)\right] - \frac{D}{2N} \left[\sum_{i=1}^{N} \cos\theta_i\right]^2 \ , \tag{27}$$

where the anisotropy term with $D > 0$ represents global coupling and favors order along the x direction. The model exhibits a transition from magnetically disordered to a magnetically ordered state along the x direction at a critical energy $\varepsilon_c = (3+D)/4$. An analysis of the Vlasov equation corresponding to this model shows that as in the isotropic case, the waterbag initial condition is stable for $\varepsilon^* < \varepsilon < \varepsilon_c$, where $\varepsilon^* = (7+D)/12$. In this energy interval a quasi-stationary state has been observed numerically, with a power law behavior (24) of the relaxation time. The exponent γ does not seem to change with the anisotropy parameter. Logarithmic growth in N of the relaxation time is found for

$\varepsilon < \varepsilon^*$. A model with local, on site anisotropy term has also been analyzed along the same lines [43]. The model is defined by the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2N}\sum_{i,j=1}^{N}(1-\cos(\theta_i-\theta_j)) + W\sum_{i=1}^{N}\cos^2\theta_i \ . \tag{28}$$

Here, too, both types of behavior have been found.

Other extensions of the HMF model include the addition of short range, nearest neighbor coupling to the Hamiltonian [33], and coupling of the HMF model to a thermal bath, making the dynamics stochastic [44]. In both cases quasi-stationarity is observed with a power law growth of the relaxation time (24) with an exponent γ which seems to vary with the interaction parameters of the models.

6. SUMMARY

Some recent statistical mechanical studies of systems with long range interactions have been reviewed. In these studies, various properties of these systems, both thermal and dynamical, have been explored within a class of models with mean-field type interactions. Models with mean-field long range interaction are non-additive, and as such they may be used to probe generic features of a wider class of systems where non-additivity plays a major role, namely, systems where the interaction between particles exhibits a power law decay with their distance. The fact that some of the properties of mean-field models may be exactly calculable, makes them particularly interesting in this context.

The canonical and microcanonical phase diagrams of a number of models have been calculated in recent studies. These include discrete spin Ising like models, such as the spin-1 Blume-Emery-Griffiths model and the Ising model with long and short range interactions, as well as continuous spin XY like models with either a fourth order global coupling or models with both long and short range interactions. The common feature found in these studies is that whenever the phase diagram exhibits a first order transition the canonical and microcanonical ensembles become non-equivalent, with the microcanonical ensemble exhibiting negative specific heat and discontinuity in temperature. Recent studies comparing the grand canonical and the canonical ensembles show that similarly, analogous differences between these ensembles are present.

Rather general considerations indicate that systems with long range interactions are likely to exhibit nonergodicity. This is a direct result of the fact that the domain in space of extensive variables, such as energy, volume and magnetization, over which the model is defined, is not necessarily convex.

Systems with long range interaction exhibit some intriguing dynamical properties. Long relaxation times of thermodynamically unstable states have been observed, with relaxation times diverging with system size N. In some cases these times diverge with a power law of N, in other cases the divergence is logarithmic with N. In some models quasi-stationary states are found, which are long lived non-equilibrium states which display unusual and intriguing properties such as anomalous diffusion and algebraically long relaxation times.

While studies of particular models provide useful insight as to the possible properties of systems with long range interactions, an overall scheme within which such properties can be classified is still missing. For example the role of various parameters (such as the symmetry and nature of the order parameter, the nature of the dynamics-whether stochastic or deterministic etc.) in determining the behavior of the system is not fully understood. It would be of great interest to use the insight obtained from studies of specific models in order to construct a more general guiding framework for these systems.

ACKNOWLEDGMENTS

I warmly thank J. Barré, F. Bouchet, P. de Buyl, A. Campa, T. Dauxois, A. Giansanti, K. Jain, R. Khomeriki, S. Ruffo and N. Schreiber for fruitful and enjoyable collaboration on systems with long range interaction over the last several years. Support of the Minerva Foundation with funding from the Federal German Ministry for Education and Research, and of the Albert Einstein Center for Theoretical Research is gratefully acknowledged.

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